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Continuous-variable and hybrid quantum gates

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Abstract

We provide several schemes to construct the continuous-variable SWAP gate and present a Hermitian generalized many-body continuous controlled^n-NOT gate. We introduce and study the hybrid controlled-NOT gate and controlled-SWAP gate, and their physical realizations are discussed in trapped-ion systems. These continuous-variable and hybrid quantum gates may be used in the corresponding continuous-variable and hybrid quantum computations.

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1. Introduction

The quantum computer [1, 2] is a device which operates with quantum logic gates. It was shown that any quantum computation can be built from a series of one-bit and two-bit quantum logic gates [3]. The fundamental controlled-NOT (CN) [4] gate, widely discussed in the literature [5], is the two-qubit gate in which one qubit is flipped conditioned on the state of another qubit. Mathematically the CN gate is defined as

$$\mathrm{CN}_{12}|i\rangle_1|j\rangle_2 = |i\rangle_1|i \oplus j\rangle_2 \tag{1}$$

where $|i\rangle_1|j\rangle_2 (i, j = 0, 1)$ are the basis states of the two qubits, $\oplus$ denotes addition modulo 2. The first (second) qubit is the control (target).

It is known that an unknown qubit state $|\psi\rangle$ can be swapped with the qubit state $|0\rangle$ using only two CN gates [6], i.e.

$$\mathrm{CN}_{21}\mathrm{CN}_{12}|\psi\rangle_1|0\rangle_2 = |0\rangle_1|\psi\rangle_2. \tag{2}$$

In [7], the gate $\mathrm{CN}_{21}\mathrm{CN}_{12}$ is called a double CN gate. Using the CN gates one can construct a general two-qubit SWAP gate as follows:

$$\mathrm{SWAP}_{12} = \mathrm{CN}_{12}\mathrm{CN}_{21}\mathrm{CN}_{12} \tag{3}$$

which makes the transformation

$$\mathrm{SWAP}_{12}|i\rangle_1|j\rangle_2 = |j\rangle_1|i\rangle_2. \tag{4}$$

The SWAP gate can be constructed in an alternative way as [8]

$$\mathrm{SWAP}_{12} = \tfrac{1}{2}(1 + \sigma_{x1}\sigma_{x2} + \sigma_{y1}\sigma_{y2} + \sigma_{z1}\sigma_{z2}) \tag{5}$$

where the operators $\sigma_{\alpha i}$ ($\alpha = x, y, z$) are the usual Pauli operators of system i. The remarkable properties of the SWAP gate are described by Collins *et al* [7], Eisert *et al* [9] and Chefles *et al* [10]. Both the CN gate and SWAP gate are two-qubit gates. The one-qubit gates include a NOT gate which is expressed by the Pauli operator σ_x and the Hadamard gate

$$H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) \tag{6}$$

which makes the transformation

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{7a}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{7b}$$

Both the NOT gate and the Hadamard gate are self-inverse, i.e. squaring them give the identity operators.

For three qubits there are two types of gates, the Toffoli gate [11] and Fredkin gate [12], which are also called the (controlled)^2-NOT gate and the controlled-SWAP (CSWAP) gate, respectively. The CSWAP gate performs the following transformation:

$$\mathrm{CSWAP}_{(12)3}|i\rangle_1|j\rangle_2|0\rangle_3 = |i\rangle_1|j\rangle_2|0\rangle_3 \tag{8a}$$

$$\mathrm{CSWAP}_{(12)3}|i\rangle_1|j\rangle_2|1\rangle_3 = |j\rangle_1|i\rangle_2|1\rangle_3 \tag{8b}$$

where the third qubit acts as the control. The quantum gates described above act on discrete variables, the qubits. In this paper we give the continuous-variable and hybrid versions of quantum gates, which may be used in continuous-variable [13] and hybrid [14] quantum computation. In the hybrid version of quantum gates the discrete variable acts as the control and the continuous variables as the targets.

In section 2 we begin with the introduction of the one-body gates for continuous variables. We proceed in section 3 to study the two-body and many-body continuous-variable gates and consider the CN gate, SWAP gate, and controlled^n-NOT gate as well as the cloning gate. Several methods are proposed to realize the SWAP gate. In section 3 we introduce and study the hybrid quantum gates, hybrid CN gates and CSWAP gates. We give two schemes to realize the hybrid gates in trapped-ion systems. The conclusion is given in section 5.

2. One-body gates for continuous variables

2.1. NOT gate

The one-body continuous-variable NOT gate may be defined as the parity operator

$$\mathrm{NOT} = (-1)^{a^\dagger a} \tag{9}$$

where a and $a^\dagger$ are bosonic annihilation and creation operators. It is easy to see that

$$\begin{aligned} \mathrm{NOT}|x\rangle &= |-x\rangle \\ \mathrm{NOT}|p\rangle &= |-p\rangle \\ \mathrm{NOT}^2 &= 1 \end{aligned} \tag{10}$$

where $|x\rangle$ is the eigenstate of the position operator $\hat{x}$, and $|p\rangle$ is the eigenstate of the momentum operator $\hat{p}$.

2.2. Hadamard gate

The continuous version of the Hadamard gate is in fact the Fourier transformation and defined by [15]

$$F(\sigma)|x\rangle = \frac{1}{\sigma\sqrt{\pi}}\int \mathrm{d}y\, \mathrm{e}^{2\mathrm{i}xy/\sigma^2}|y\rangle \tag{11}$$

where σ is the scaled length. This is the transformation used to go from the position to the momentum basis if we set $\sigma = \sqrt{2}$. The inverse $F^{\dagger}(\sigma)$ is obtained by replacing i by $-i$ giving the result that

$$F(\sigma)F^{\dagger}(\sigma)|x\rangle = F^{\dagger}(\sigma)F(\sigma)|x\rangle = |x\rangle. \tag{12}$$

Note that the continuous-variable Hadamard gate is not self-inverse.

3. Two-body and many-body gates for continuous variables

3.1. CN gate

The two-qubit CN gate has been extended to the case of continuous variables, the gates CN_{12}^{+} [15] and CN_{12}^{-} [16], which are defined by

$$\mathrm{CN}_{12}^{\pm}|x\rangle_1|y\rangle_2 = |x\rangle_1|x \pm y\rangle_2 \tag{13}$$

$$\mathrm{CN}_{12}^{+} = \mathrm{e}^{-\mathrm{i}\hat{x}_1\hat{p}_2} \tag{14}$$

$$\mathrm{CN}_{12}^{-} = \mathrm{NOT}_2\mathrm{e}^{\mathrm{i}\hat{x}_1\hat{p}_2} = \mathrm{e}^{-\mathrm{i}\hat{x}_1\hat{p}_2}\mathrm{NOT}_2 \tag{15}$$

where the position operator of system i $(i = 1, 2)$ is denoted by $\hat{x}_i$ and the momentum operator by $\hat{p}_i$. In momentum space the CN gate can be defined as

$$\mathrm{CN}_{12}^{\pm}|p\rangle_1|q\rangle_2 = |p\rangle_1|p \pm q\rangle_2 \tag{16}$$

$$\mathrm{CN}_{12}^{+} = \mathrm{e}^{\mathrm{i}\hat{x}_2\hat{p}_1} \tag{17}$$

$$\mathrm{CN}_{12}^{-} = \mathrm{NOT}_2\mathrm{e}^{-\mathrm{i}\hat{x}_2\hat{p}_1} = \mathrm{e}^{\mathrm{i}\hat{x}_2\hat{p}_1}\mathrm{NOT}_2. \tag{18}$$

The definitions of the CN gates are basis dependent. From equations (14) and (15), it is easy to check that both gates are unitary, the gate CN_{12}^{+} is not Hermitian and not self-inverse, while CN_{12}^{-} is Hermitian and self-inverse.

The CN gate for qubits has been used in various kinds of quantum information processing such as teleportation [17], dense coding [18], quantum state swapping [4], entangling quantum states [19] and Bell measurements [20]. It is natural to ask that if the continuous CN gates can perform some similar tasks like entangling and swapping quantum states. Let the continuous CN gates $\mathrm{CN}_{12}^{\pm}$ and the Hadamard gate $F(\sqrt{2})$ act on the state $|z\rangle_1|y\rangle_2$. The resultant states are entangled states:

$$|\psi\rangle^{\pm} = \mathrm{CN}_{12}^{\pm}F(\sqrt{2})|z\rangle_1|y\rangle_2 = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}x\, \mathrm{e}^{\mathrm{i}xz}|x\rangle_1|x \pm y\rangle_2. \tag{19}$$

It is interesting to see that the following equations:

$$(\hat{x}_1 - \hat{x}_2)|\psi\rangle^{\pm} = \mp y|\psi\rangle^{\pm} \tag{20a}$$

$$(\hat{p}_1 + \hat{p}_2)|\psi\rangle^{\pm} = z|\psi\rangle^{\pm} \tag{20b}$$

hold. That is to say, both the entangled states $|\psi\rangle^{\pm}$ are the common eigenvectors of the position difference operator $\hat{x}_1 - \hat{x}_2$ and momentum sum operator $\hat{p}_1 + \hat{p}_2$. Furthermore, both the continuous CN gates can be used to construct the N-party entangled state as follows:

$$\mathrm{CN}_{12}^{\pm}\mathrm{CN}_{13}^{\pm}\ldots\mathrm{CN}_{1N}^{\pm}|p=0\rangle_1|x=0\rangle_2|x=0\rangle_3\ldots|x=0\rangle_N = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}x\,|x\rangle_1|x\rangle_2\ldots|x\rangle_N. \tag{21}$$

This state is obtained by Braunstein [15] by a series of beam splitters. Here we provide an alternative way to obtain this state by using N CN gates. The N-party entangled state is an eigenstate with total momentum zero and relative positions zero.

3.2. SWAP gate

Having seen that both the continuous CN gates can entangle quantum states, then we ask if they can perform quantum state swapping by certain combinations of them. For continuous variables we have

$$\mathrm{CN}_{21}^{-}\mathrm{CN}_{12}^{\pm}|x\rangle_1|y=0\rangle_2 = |y=0\rangle_1|x\rangle_2. \tag{22}$$

From equation (3), one may guess that a similar expression exists for a continuous-variable SWAP gate. It is straightforward to check that

$$\mathrm{CN}_{12}^{+}\mathrm{CN}_{21}^{+}\mathrm{CN}_{12}^{+}|x\rangle_1|y\rangle_2 = |2x+y\rangle_1|3x+2y\rangle_2 \tag{23a}$$

$$\mathrm{CN}_{12}^{-}\mathrm{CN}_{21}^{-}\mathrm{CN}_{12}^{-}|x\rangle_1|y\rangle_2 = |-y\rangle_1|-x\rangle_2. \tag{23b}$$

Then the SWAP gate can be constructed as

$$\mathrm{SWAP}_{12} = \mathrm{NOT}_1\mathrm{NOT}_2\mathrm{CN}_{12}^{-}\mathrm{CN}_{21}^{-}\mathrm{CN}_{12}^{-} = \mathrm{CN}_{12}^{-}\mathrm{CN}_{21}^{-}\mathrm{CN}_{12}^{-}\mathrm{NOT}_1\mathrm{NOT}_2 \tag{24}$$

$$\mathrm{SWAP}_{12}|x\rangle_1|y\rangle_2 = |y\rangle_1|x\rangle_2. \tag{25}$$

We see that one cannot obtain the SWAP gate using only the CN_{ij}^{+} $(i \neq j)$ gates , while one can use the CN_{ij}^{-} gates to obtain it. Different from the situation of discrete variables, here the continuous-variable SWAP gate needs two NOT gates. In fact, the gate CN_{ij}^{+} $(i \neq j)$ is not completely useless in the realization of the SWAP gate. Using both the CN_{ij}^{+} and CN_{ij}^{-} gates, we have

$$\mathrm{SWAP}_{12} = \mathrm{NOT}_2\mathrm{CN}_{12}^{-}\mathrm{CN}_{21}^{-}\mathrm{CN}_{12}^{+} = \mathrm{e}^{\mathrm{i}\hat{x}_1\hat{p}_2}\mathrm{NOT}_1\mathrm{e}^{\mathrm{i}\hat{x}_2\hat{p}_1}\mathrm{e}^{-\mathrm{i}\hat{x}_1\hat{p}_2}. \tag{26}$$

Here we have used equations (14) and (15). Then we can construct the SWAP gate using the one-body gates and three two-body gates. The SWAP gate acting on momentum space can be constructed similarly.

Recalling that the two-qubit SWAP gate can be given in equation (5), we expect that the continuous SWAP gate be implemented in another way. Now we introduce the operator

$$B_{12} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\hat{x}_1\hat{p}_2-\hat{x}_2\hat{p}_1)} \tag{27}$$

acting on the two continuous systems 1 and 2. The operator corresponds to a beam splitter and performs the transformation

$$B_{12}\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}B_{12}^{\dagger} = \begin{pmatrix}-\hat{p}_2\\ \hat{p}_1\end{pmatrix} \tag{28}$$

from which we have

$$B_{12}|x\rangle_1|y\rangle_2 = |y\rangle_1|-x\rangle_2. \tag{29}$$

Then the continuous-variable SWAP gate is immediately obtained as

$$\mathrm{SWAP}_{12} = \mathrm{NOT}_2 B_{12}. \tag{30}$$

From equations (28) and (30), the swapping function of the SWAP gate can be compactly stated by

$$\begin{aligned} \mathrm{SWAP}_{12}\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}\mathrm{SWAP}_{12} &= \begin{pmatrix}\hat{p}_2\\ \hat{p}_1\end{pmatrix} \\ \mathrm{SWAP}_{12}\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix}\mathrm{SWAP}_{12} &= \begin{pmatrix}\hat{x}_2\\ \hat{x}_1\end{pmatrix} \end{aligned} \tag{31}$$

which may serve as alternative definitions.

Substituting $\hat{x}_j = \frac{1}{\sqrt{2}}(a_j + a_j^\dagger)$, $\hat{p}_j = \frac{1}{\mathrm{i}\sqrt{2}}(a_j - a_j^\dagger)$ into equation (27), we can re-express the operator B_{12} in terms of the annihilation and creation operators and then rewrite the SWAP gate (30) as

$$\mathrm{SWAP}_{12} = \mathrm{e}^{\mathrm{i}\pi a_2^\dagger a_2}\mathrm{e}^{\frac{\pi}{2}(a_1^\dagger a_2 - a_2^\dagger a_1)}. \tag{32}$$

Letting the above SWAP gate act on the discrete Fock basis states, we obtain

$$\mathrm{SWAP}_{12}|n\rangle_1|m\rangle_2 = |m\rangle_1|n\rangle_2 \tag{33}$$

where $|n\rangle_i$ denotes the Fock state of system i. In fact, equation (33) gives the representation of the SWAP gate in the two-mode Fock space. We see that the SWAP gate is basis independent, while the CN gate is basis dependent.

To conclude this section we mention a relation between the SWAP gate and the CN gates:

$$\mathrm{SWAP}_{12}\mathrm{CN}_{12}\mathrm{SWAP}_{12} = \mathrm{CN}_{21}. \tag{34}$$

The above equation shows that one can use the SWAP gate and CN gate CN_{12} to realize another CN gate CN_{21}.

3.3. *Controlled^n-NOT gate*

We define a Hermitian continuous generalization of the discrete controlled^n-NOT gate as

$$\mathrm{CN}_{(12\ldots N)N+1}|x_1\rangle_1|x_2\rangle_2\ldots|x_N\rangle_N|x_{N+1}\rangle_{N+1} = |x_1\rangle_1|x_2\rangle_2\ldots|x_N\rangle_N\left|-x_{N+1} + \sum_{n=1}^{N} x_n\right\rangle_{N+1} \tag{35}$$

$$\mathrm{CN}_{(12\ldots N)N+1} = \mathrm{NOT}_{N+1}\exp\left(\mathrm{i}\hat{p}_{N+1}\sum_{n=1}^{N}\hat{x}_n\right). \tag{36}$$

A similar gate can be defined in momentum space. Then the gate defined in this way is both unitary and Hermitian, and therefore self-inverse. For the $N = 2$ and 3, cases the gate becomes the continuous-variable CN and Toffoli gates, respectively.

3.4. $1 \to 2$ *cloning gate*

For discrete variables the CN gates CN_{21} and CN_{31} commute with each other, however, for continuous variables, from equation (15) the following equation:

$$[\mathrm{CN}_{31}^-, \mathrm{CN}_{21}^-] = \mathrm{e}^{\mathrm{i}(\hat{x}_2 - \hat{x}_3)\hat{p}_1} - \mathrm{e}^{\mathrm{i}(\hat{x}_3 - \hat{x}_2)\hat{p}_1} \tag{37}$$

holds for two Hermitian CN gates CN_{21}^- and CN_{31}^-. That is to say, these two continuous-variable CN gates do not commute.

It is known that the $1 \to 2$ cloning gate is described by [21]

$$\mathcal{C} = \mathrm{CN}_{31}\mathrm{CN}_{21}\mathrm{CN}_{13}\mathrm{CN}_{12} \tag{38}$$

in terms of four CN gates. To generalize directly to the continuous case of the above cloning gate, we obtain

$$\mathcal{C}' = \mathrm{CN}_{31}^-\mathrm{CN}_{21}^-\mathrm{CN}_{13}^-\mathrm{CN}_{12}^-. \tag{39}$$

Using equations (15) and (37), we rewrite the gate $\mathcal{C}'$ as

$$\mathcal{C}' = \mathrm{e}^{-\mathrm{i}(\hat{x}_3-\hat{x}_2)\hat{p}_1}\mathrm{e}^{-\mathrm{i}\hat{x}_1(\hat{p}_2+\hat{p}_3)}\mathrm{NOT}_2\mathrm{NOT}_3 \tag{40}$$

which is just the continuous-variable $1 \to 2$ cloning gate up to the two NOT gates [22].

4. Hybrid gates

Now we introduce and study two kinds of hybrid quantum gates, the hybrid CN gate and CSWAP gate.

4.1. Hybrid CN gate

We define the hybrid CN gate as

$$\begin{aligned}\mathrm{CN}'_{12}|0\rangle_1|x\rangle_2 &= |0\rangle_1|x\rangle_2\\ \mathrm{CN}'_{12}|1\rangle_1|x\rangle_2 &= |1\rangle_1|-x\rangle_2\end{aligned}$$

which can be realized in a trapped-ion system. In trapped-ion systems, one can have the following Hamiltonian experimentally [23, 24]:

$$H_1 = \lambda a^\dagger a \mathcal{P}_1 \tag{41}$$

where a and $a^\dagger$ are bosonic annihilation and creation operators of the centre-of-mass motion of the trapped ion, $\mathcal{P}_1 = |1\rangle_1\langle 1|$ is the projection operator and λ is the effective coupling constant. It is easy to show that the evolution operator $\mathrm{e}^{-\mathrm{i}\lambda t a^\dagger a \mathcal{P}_1}$ at time $t = \pi/\lambda$ gives directly the hybrid CN gate. One simple application of this gate is the generation of even and odd coherent states. Let the input state be $\frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)|\alpha\rangle_2$, where $|\alpha\rangle_2$ is a bosonic coherent state. Then after the gate operation the output state will be $\frac{1}{\sqrt{2}}(|0\rangle_1|\alpha\rangle_2 + |1\rangle_1|-\alpha\rangle_2)$. Now we measure the qubit on the state $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, the continuous state will collapse into the even and odd coherent states, respectively.

4.2. Hybrid controlled-SWAP gate

A general controlled-SWAP gate is described by the following transformation:

$$\begin{aligned}|\Psi\rangle_1|\Phi\rangle_2|0\rangle_3 &\to |\Psi\rangle_2|\Phi\rangle_1|0\rangle_3\\ |\Psi\rangle_1|\Phi\rangle_2|1\rangle_3 &\to |\Phi\rangle_2|\Psi\rangle_1|1\rangle_3.\end{aligned} \tag{42}$$

This gate has three inputs and the third is the control qubit. Let the input state of the CSWAP gate be $\frac{1}{\sqrt{2}}|\Psi\rangle_1|\Phi\rangle_2(|0\rangle_3 + |1\rangle_3)$ and measure the output state. If we measure the qubit on the state $|\pm\rangle_3 = \frac{1}{\sqrt{2}}(|0\rangle_3 \pm |1\rangle_3)$, we obtain exactly the symmetric and antisymmetric entangled states, $|\Psi\rangle_1|\Phi\rangle_2 \pm |\Phi\rangle_2|\Psi\rangle_1$ up to normalization constants. This is actually a universal entangler [25]. So it is desirable to consider the CSWAP gate of the form (42) when the states $|\Psi\rangle_1$ and $|\Phi\rangle_2$ are continuous-variable states.

From the continuous-variable SWAP gate (26), the CSWAP gate is formally constructed as

$$\mathrm{CSWAP}'_{12(3)} = \mathrm{e}^{\mathrm{i}\hat{x}_1\hat{p}_2\mathcal{P}_3}\mathrm{e}^{\mathrm{i}\pi a_1^\dagger a_1\mathcal{P}_3}\mathrm{e}^{\mathrm{i}\hat{x}_2\hat{p}_1\mathcal{P}_3}\mathrm{e}^{-\mathrm{i}\hat{x}_1\hat{p}_2\mathcal{P}_3} \tag{43}$$

where $\mathcal{P}_3 = |1\rangle_3\langle 1|$ is the projection operator of control system 3. There are three three-body interactions in the expression of the CSWAP gate. We will realize the CSWAP gate by two-body interactions.

First we see that the operators $e^{\pm i x\hat{p}}$ and $e^{\pm i p\hat{x}}$ satisfy the relation

$$e^{ixp} = e^{ix\hat{p}}e^{ip\hat{x}}e^{-ix\hat{p}}e^{-ip\hat{x}}. \tag{44}$$

The above relation can be generalized as [26]

$$e^{ixp\sin\theta} = e^{i(\frac{\pi}{2}-\theta)a^\dagger a}e^{ix\hat{p}}e^{-i(\frac{\pi}{2}-\theta)a^\dagger a}e^{ip\hat{x}}e^{i(\frac{\pi}{2}-\theta)a^\dagger a}e^{-ix\hat{p}}e^{-i(\frac{\pi}{2}-\theta)a^\dagger a}e^{-ip\hat{x}}. \tag{45}$$

As the operator $\hat{p}_1$, $\hat{x}_2$ and $\mathcal{P}_3$ commutes with each other, we replace x with $\hat{x}_2$, p with $\hat{p}_1$, and θ with $\pi\mathcal{P}_3/2$ in equation (45), respectively. Then we obtain

$$e^{i\hat{p}_1\hat{x}_2\mathcal{P}_3} = e^{i\frac{\pi}{2}(1-\mathcal{P}_3)a^\dagger a}e^{i\hat{x}_2\hat{p}}e^{-i\frac{\pi}{2}(1-\mathcal{P}_3)a^\dagger a}e^{i\hat{p}_1\hat{x}}e^{i\frac{\pi}{2}(1-\mathcal{P}_3)a^\dagger a}e^{-i\hat{x}_2\hat{p}}e^{-i\frac{\pi}{2}(1-\mathcal{P}_3)a^\dagger a}e^{-i\hat{p}_1\hat{x}}. \tag{46}$$

The above equation shows that we have written the three-body unitary operator $e^{i\hat{p}_1\hat{x}_2\mathcal{P}_3}$ in terms of eight two-body operators. Therefore the CSWAP gate (43) can be written in terms of two-body operators.

From equations (27) and (30), we write the CSWAP gate in the form

$$\mathrm{CSWAP}'_{12} = e^{i\pi a_2^\dagger a_2\mathcal{P}_3}e^{i\frac{\pi}{2}(\hat{x}_1\hat{p}_2-\hat{x}_2\hat{p}_1)\mathcal{P}_3} \tag{47}$$

which also includes a three-body operator. Next we see how to realize this CSWAP gate in a trapped-ion system.

Gerry derived an effective Hamiltonian for two modes a and b as [27]

$$H_2 = \chi(a_1^\dagger a_1 - a_2^\dagger a_2)\mathcal{P}_3 \tag{48}$$

in a trapped-ion system. The Hamiltonian H_2 can be rewritten as

$$H_2 = 2\chi J_z\mathcal{P}_3 \tag{49}$$

where $J_z = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$. The operators J_z, $J_+ = a_1^\dagger a_2$, and $J_- = a_2^\dagger a_1$ form the $su(2)$ Lie algebra. The unitary operator at time $t = -\pi/(2\chi)$ corresponding to the Hamiltonian is given by

$$U = e^{i\pi J_z\mathcal{P}_3}. \tag{50}$$

The unitary operator U can be transformed to U' as

$$\begin{aligned} U' &= e^{i\frac{\pi}{2}J_x}Ue^{-i\frac{\pi}{2}J_x} \\ &= e^{i\pi J_y\mathcal{P}_3} = e^{i\frac{\pi}{2}(\hat{x}_1\hat{p}_2-\hat{x}_2\hat{p}_1)\mathcal{P}_3} \end{aligned} \tag{51}$$

where $J_x = (J_+ + J_-)/2$ and $J_y = (J_+ - J_-)/(2i)$. From equations (47) and (51), we write the CSWAP gate as

$$\begin{aligned} \text{C-SWAP}'_{12} &= e^{i\pi a_2^\dagger a_2\mathcal{P}_3}e^{i\frac{\pi}{4}(a_1^\dagger a_2+a_2^\dagger a_1)}e^{i\frac{\pi}{2}a_1^\dagger a_1\mathcal{P}_3}e^{-i\frac{\pi}{2}a_2^\dagger a_2\mathcal{P}_3}e^{-i\frac{\pi}{4}(a_1^\dagger a_2+a_2^\dagger a_1)} \\ &= e^{i\frac{\pi}{2}a_2^\dagger a_2\mathcal{P}_3}e^{-i\frac{\pi}{2}a_1^\dagger a_1\mathcal{P}_3}e^{i\frac{\pi}{4}(a_1^\dagger a_2+a_2^\dagger a_1)}e^{i\pi a_1^\dagger a_1\mathcal{P}_3}e^{-i\frac{\pi}{4}(a_1^\dagger a_2+a_2^\dagger a_1)}. \end{aligned} \tag{52}$$

Therefore we have given a form of CSWAP gate in terms of five two-body operators.

We have used two methods to express the three-body hybrid CSWAP gate in terms of two-body operators. In other words we provide two ways to realize the CSWAP gate.

5. Conclusion

In conclusion we have introduced and studied the continuous and hybrid versions of quantum gates. The continuous-variable gates include one-body (NOT, Hadamard), two-body (CN, double CN, SWAP) and many-body gates (controlled^n-NOT). Some relations between the CN, double CN and the SWAP gates are given. The hybrid quantum gates include the hybrid CN gate and the three-body controlled-SWAP gate. We proposed physical schemes to realize the hybrid gates in the trapped-ion systems. It is interesting to see that most of the quantum gates are not only unitary, but also Hermitian, and therefore self-inverse.

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